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Brane transfer operations and T-duality of non-BPS states

Oren Bergman, Eric G. Gimon and Petr Hořava

California Institute of Technology,
Pasadena, CA 91125, USA
E-mail: bergman@theory.caltech.edu, egimon@theory.caltech.edu, horava@theory.caltech.edu

Abstract: Using the relation between D-brane charges and K-theory, we study non-BPS D-branes and their behavior under T-duality. We point out that in general compactifications, D-brane charges are classified by relative K-theory groups. T-duality is found to act as a symmetry between the relative K-theory groups in Type II and Type I/IA theories. We also study Type $\widetilde{\mathrm{IA}}$ theory (which contains an $\mathrm{O8}^-$ plane and an $\mathrm{O8}^+$ plane), using K-theory and T-duality to identify its stable D-branes. Comparison with string theory constructions reveals two interesting effects. One of them involves the transfer of branes between O-planes, while in the other, a D-brane charge which seems conserved near one O-plane in fact decays due to the presence of another type of O-plane.

Keywords: Superstrings and Heterotic Strings, D-branes.

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Contents

1. Introduction

A closer look at the dynamics of various unstable D-brane systems (such as brane-antibrane systems) has recently opened a new perspective for understanding D-branes and their conserved charges. Traditionally, D-branes are understood as RR-charged stringy solitons on which strings can end; in the new framework [1, 2, 3, 4], D-branes appear as topological defects in the worldvolume of unstable brane systems of higher dimension.

A crucial role in this construction is played by our improved understanding of the string theory tachyon [1] (see also [5]–[6]). String theory has been plagued with tachyons since its early days, but whether they represent an incurable instability of the theory or have a more subtle role in the dynamics was not known. It is now believed that the tachyonic mode of the open string stretching between a D-brane and a D-antibrane (or between a pair of unstable D-branes) is really a legitimate Higgs field, and therefore does not represent an incurable instability. Instead, it tends to develop a stable vacuum expectation value, leading to the decay of the unstable state into a stable state. Depending on the details of the original unstable configuration, the resulting stable state can contain topological defects that correspond to stable D-branes.

Any such construction can be related through a hierarchy of embeddings to bound states in the unstable system of a number of spacetime-filling D9-branes. The worldvolume dynamics of this system contains $\mathrm{U}(N) \times \mathrm{U}(N)$ Yang-Mills theory and a Higgs field (a.k.a. "tachyon") in the $(\mathbf{N}, \overline{\mathbf{N}})$ representation in the case of Type IIB theory [1][2], and $\mathrm{U}(N)$ Yang-Mills theory with an adjoint Higgs in the Type IIA case [3]. All possible stable D-branes — both supersymmetric *and* non-supersymmetric — appear as topological defects in the worldvolume Higgs field on these spacetime-filling D-brane systems. In this sense, the spacetime-filling brane system provides a universal medium in which all stable D-brane charges are carried by conventional topological defects, similar to vortices in Type II superconductors or magnetic monopoles of grand unified theories.

The precise dynamics of these unstable D-brane systems is not known, but the topological information needed for the complete classification of D-brane charges can still be determined. This information is usefully encoded in K-theory [2, 3, 7]. (The connection between D-brane charges and K-theory was first suggested [8].) Once one identifies the K-theory group relevant to a given compactification, one can use methods developed in the mathematical literature to compute it, and thereby determine the spectrum of conserved D-brane charges. Having classified the charges, one can then look for a string theory construction of the corresponding D-branes.

One of the defining qualities of D-branes, which in fact is how they were discovered [9, 10], is their transformation under T-duality. Because T-duality exchanges Neumann open string boundary conditions with Dirichlet ones, it exchanges wrapped branes and unwrapped branes. One of the purposes of this paper is to determine how T-duality is manifested in the bound state construction above. We shall therefore study compactifications on $X \times \mathbf{S}^1$, and orientifolds thereof.

In section 2 we discuss the K-theory realization of T-duality in Type II string theory. As one of the central points of the paper, we show that in general string theory compactifications, D-brane charges are classified by *relative* K-theory groups, such as $\mathrm{K}(\mathbf{S}^p \times Y, Y)$, where Y is the compactification manifold. In the case of Type II strings compactified on a circle, the relative K-theory group of D-brane charges splits into the sum of two groups, whose elements reflect the split of the D-brane charges between wrapped D-branes and unwrapped D-branes. The K-group for IIB on a circle, $\mathrm{K}(X \times \mathbf{S}^1, \mathbf{S}^1)$, is isomorphic to the K-group for IIA on a circle, $\mathrm{K}^{-1}(X \times \mathbf{S}^1, \mathbf{S}^1)$, clearly in line with T-duality. Moreover, the isomorphism exchanges the subgroup associated with wrapped D-branes on one side with the subgroup for unwrapped D-branes on the other side. In the rest of this paper, we would like to see if this clear split between wrapped and unwrapped D-branes holds generically in K-theory.

Our first test case, discussed in section 3, involves looking at the T-duality between Type I strings on a circle and the Type IA orientifold with two $O8^-$ orientifold planes. Unlike in the Type II theory, there are now new non-BPS D-branes with $\mathbb{Z}_2$ valued charges. At first, the situation looks quite similar to the Type II case: The K-group for Type I on a circle, $\mathrm{KO}(X \times \mathbf{S}^1, \mathbf{S}^1)$, is again found to be isomorphic to the Type IA K-group $\mathrm{KR}^{-1}(X \times \mathbf{S}^1, \mathbf{S}^1)$, and it splits into two parts which one may naively interpret as corresponding to wrapped branes and unwrapped branes.

This interpretation raises two puzzles. First, some of the non-BPS D-branes that were stable in flat space are now only stable for a certain range of the circle's radius, even though they carry a conserved charge. Second, in Type IA we expect to find unwrapped branes localized on each of the two orientifold planes, yet the corresponding charges seem to be missing from the subgroup we naively associate with unwrapped branes. Section 3 will demonstrate how these two puzzles are resolved. A key element in this resolution involves processes that we refer to as brane transfer operations.

Our second test case, presented in section 4, deals with the more exotic Type $\widetilde{\mathrm{I}}$ open string theory, which is T-dual to the Type $\widetilde{\mathrm{IA}}$ orientifold with both an $O8^-$ and an $O8^+$ orientifold plane. The K-group associated with these theories, which turns out to be equivalent to a certain K-theory group known in the mathematical literature as $\widetilde{\mathrm{KSC}}(X)$, does not split naturally into the of sum of two sub-groups. The split between wrapped branes and unwrapped branes is totally defeated. We will show that the root of this problem is linked to the fact that some non-BPS brane configurations locally stable at the $O8^-$ orientifold plane become unstable due to the presence of the $O8^+$ plane and vice versa. This phenomenon can have important consequences for the piecewise analysis of the stable non-BPS D-brane spectra in various compactifications, such as when one approximates singularities in K3 orientifolds by ALE spaces [11].

Many technical details required for our analysis have been relegated to an appendix, which also serves as a collection of basic facts in K-theory, and can therefore be of some independent interest.

While this paper was being written, two papers [12, 13] appeared in which some overlapping results on T-duality in K-theory were obtained. The connection between KR-theory and orientifolds has also been discussed in [14].

2. Type II theories

As a warm-up exercise, we set the stage for our later analysis of stable D-branes in various orientifold models by first analyzing the case of Type II theories compactified on a circle to nine dimensions. (As we will see, this procedure can be easily iterated to understand $\mathbf{T}^n$ compactifications). It turns out that all stable D-brane states predicted by K-theory in these compactifications carry conventional Ramond-Ramond charges. Therefore, we do not expect any surprises; the main goal in this brief section is to see that T-duality of Type II theories is indeed a manifest symmetry in K-theory.

2.1 $D = 10$

Stable Dp-brane charges in Type IIA and Type IIB theory on $\mathbb{R}^{10}$ are classified by the K-theory groups $\mathrm{K}^{-1}(\mathbf{S}^{9-p})$ and $\widetilde{\mathrm{K}}(\mathbf{S}^{9-p})$ respectively, where the sphere $\mathbf{S}^{9-p}$ represents the dimensions transverse to the worldvolume of the p-brane, compactified by adding a point at infinity. This result can be derived by realizing supersymmetric Type II D-branes as stable topological defects in the Higgs field on the worldvolume of a system of spacetime-filling D9-branes [2, 3]. Central to this derivation is the relation (reviewed in the appendix) between homotopy theory, which classifies topological defects, and K-theory, which classifies configurations of spacetime-filling branes up to creation and annihilation. According to this relation, the K-theory groups of spheres are equal to the homotopy groups of the vacuum manifold of the Higgs field that appears in the worldvolume of the corresponding system of spacetime-filling branes. In Type IIB theory [2], the Higgs field is in the $(\mathbf{N}, \overline{\mathbf{N}})$ of the $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge group, and its vacuum manifold is a copy of $\mathrm{U}(N)$. Its homotopy groups are related to K-theory via

$$\widetilde{\mathrm{K}}(\mathbf{S}^n) = \pi_{n-1}(\mathrm{U}(N)). \tag{2.1}$$

In Type IIA theory [3], the Higgs field is in the adjoint of the $\mathrm{U}(2N)$ gauge group, and the vacuum manifold is given by the group coset $\mathrm{U}(2N)/\mathrm{U}(N) \times \mathrm{U}(N)$. This is in turn related to K-theory groups by

$$\mathrm{K}^{-1}(\mathbf{S}^n) = \pi_{n-1}\left(\mathrm{U}(2N)/\mathrm{U}(N) \times \mathrm{U}(N)\right). \tag{2.2}$$

These K-theory groups, shown in the table below, reproduce the known spectrum of BPS Dp-branes in the Type II theories in $\mathbb{R}^{10}$, with $p = 0, 2, 4, 6, 8$ in Type IIA and $p = -1, 1, 3, 5, 7$ in Type IIB.[1]

Dp-brane	D9	D8	D7	D6	D5	D4	D3	D2	D1	D0	D(-1)
Transverse $\mathbf{X}$	$\mathbf{S}^0$	$\mathbf{S}^1$	$\mathbf{S}^2$	$\mathbf{S}^3$	$\mathbf{S}^4$	$\mathbf{S}^5$	$\mathbf{S}^6$	$\mathbf{S}^7$	$\mathbf{S}^8$	$\mathbf{S}^9$	$\mathbf{S}^{10}$
$\widetilde{\mathrm{K}}(\mathbf{X})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\mathrm{K}^{-1}(\mathbf{X})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Table 1: K^{-1} and $\widetilde{\mathrm{K}}$ groups.

[1] We have included the charge of the spacetime-filling Type IIA D9-branes in the table (and will do so systematically in similar cases throughout the paper), even though the absence of a net 9-brane charge in Type IIB theory is forced by the condition of tadpole anomaly cancellation.

2.2 General compactifications

The next step would be to consider Type II theory compactified on $\mathbf{S}^1$. Before we discuss this case in detail, it seems worthwhile to first study the classification of D-brane charges for a more general compactification space Y of dimension d. We are interested in finding all D-brane charges of codimension n in the non-compact space $\mathbb{R}^{9-d}$. Such charges will arise both from D-branes located at particular points in Y, and from D-branes which wrap non-trivial cycles in Y. Since we are only interested in objects of finite energy (or action), we consider only configurations that are equivalent to the vacuum asymptotically in the transverse space $\mathbb{R}^n$, i.e. along a copy of the entire compactification manifold Y at infinity. Therefore $\mathbb{R}^n$ is effectively replaced with $\mathbf{S}^n$ by adding a point at infinity, which corresponds in the full theory to adding a copy of the compactification manifold Y at infinity. In mathematical terms, this requires us to consider bundles which are trivialized on the compactification manifold Y at infinity; such bundles define groups known in the mathematical literature as *relative* K-theory groups (cf. the appendix). Thus, we conclude that the proper way of understanding the spectrum of D-brane charges is in terms of relative K-theory. In Type IIB and Type IIA theory on Y, the relative groups that classify D-brane charges are denoted by $\mathrm{K}(\mathbf{S}^n \times Y, Y)$ and $\mathrm{K}^{-1}(\mathbf{S}^n \times Y, Y)$, respectively.

The argument leading to the appearance of relative K-theory groups is essentially independent of the type of string theory considered. It suggests the following prescription for identifying stable D-brane charges in general string theory compactifications on $\mathbb{R}^{9-d} \times Y$, at least when no RR backgrounds or non-trivial $B_{\mu\nu}$ backgrounds are excited: *Stable charges carried by D-branes of codimension n in the non-compact dimensions are classified by the relative K-theory groups $\mathcal{K}^{-q}(\mathbf{S}^n \times Y, Y)$.* The value of q and the type $\mathcal{K}$ of K-theory depends on the type of string theory and the compactification manifold Y.

2.3 $D = 9$

Having clarified the appearance of relative K-theory groups in the classification of D-brane charges in general compactifications, we can now return to Type II theory on a circle. Using (A.9) and arguments presented in the appendix, the relative groups that classify D-brane charges in Type IIB and Type IIA theory on $\mathbf{S}^1$ can be shown to decompose as follows:

$$\mathrm{K}(X \times \mathbf{S}^1, \mathbf{S}^1) = \mathrm{K}^{-1}(X) \oplus \widetilde{\mathrm{K}}(X)\,, \tag{2.3}$$

and

$$\mathrm{K}^{-1}(X \times \mathbf{S}^1, \mathbf{S}^1) = \widetilde{\mathrm{K}}^{-2}(X) \oplus \mathrm{K}^{-1}(X)\,, \tag{2.4}$$

where in both cases the first term is the contribution of unwrapped branes to the nine-dimensional D-brane charge, and the second term is the contribution of wrapped branes. Since by Bott periodicity $\widetilde{\mathrm{K}}^{-2}(X) \cong \widetilde{\mathrm{K}}(X)$, the above groups are isomorphic

$$\mathrm{K}(X \times \mathbf{S}^1, \mathbf{S}^1) \cong \mathrm{K}^{-1}(X \times \mathbf{S}^1, \mathbf{S}^1)\,, \tag{2.5}$$

and we recover the result that the spectrum of D-brane charges in nine dimensions is identical for Type IIA and Type IIB. In fact, for each $X = \mathbf{S}^n$, the relative K-theory group of D-brane charges is $\mathbb{Z}$ — the RR charge of the corresponding Dp-brane.

Furthermore, since the above isomorphism maps the first (second) term in (2.3) to the second (first) term in (2.4), and therefore exchanges unwrapped and wrapped D-branes, it corresponds precisely to T-duality. More rigorously, this follows from a derivation of (2.5) that keeps track of the multiplicative structure of K-theory (see (A.12) of the appendix and the discussion therein).

2.4 $D < 9$

We can iterate the steps of the previous subsection, and extend our results to higher toroidal compactifications. Thus, the relative group of D-brane charges in Type IIB theory is

$$\mathrm{K}(X \times \mathbf{T}^m,\, \mathbf{T}^m) = \bigoplus_{n=0}^{m} \binom{m}{n} \widetilde{\mathrm{K}}^{-n}(X) = \widetilde{\mathrm{K}}(X)^{\oplus 2^{m-1}} \oplus \mathrm{K}^{-1}(X)^{\oplus 2^{m-1}}\,, \tag{2.6}$$

where the second equality follows by Bott periodicity. An analogous calculation on the Type IIA side (cf. the appendix) reveals

$$\mathrm{K}^{-1}(X \times \mathbf{T}^m,\, \mathbf{T}^m) \cong \mathrm{K}(X \times \mathbf{T}^m,\, \mathbf{T}^m)\,. \tag{2.7}$$

This proves T-duality of D-brane charges in Type II theory on $\mathbf{T}^m$, and gives the expected degeneracy of Dp-brane charges arising from wrapping all higher supersymmetric branes on various cycles of the torus. All in all, this shows that in the case of Type II theory on $\mathbf{T}^m$, D-brane charges predicted by the more precise K-theory arguments coincide with those predicted by the somewhat cruder argument that relates D-brane charges to RR charges (and therefore to the cohomology of the compactification manifold).

3. Type I Theory and its T-Duals

Our next case study is Type I string theory. Here we encounter two new features: $\mathbb{Z}_2$-charged non-BPS D-branes, and discrete $\mathbb{Z}_2$-valued Wilson lines on some of the wrapped D-branes. The latter would seem to require additional $\mathbb{Z}_2$ charges in the D-brane spectrum, naively absent in K-theory. Surprisingly, we shall see that the $\mathbb{Z}_2$ charges of unwrapped non-BPS D-branes already incorporate the $\mathbb{Z}_2$ Wilson lines of wrapped D-branes. In T-dual orientifolds this is seen via "brane transfer operations", whereby an unwrapped brane at one orientifold plane is "transferred" by a wrapped brane to another orientifold plane.

3.1 D=10

In Type I theory, the full spectrum of D-brane charges can be determined from the dynamics of unstable systems of multiple D9-brane $\overline{\text{D9}}$-brane pairs. Since the action of the orientifold group is antilinear on Chan-Paton bundles, the K-theory that arises in such systems is the KO-theory of real virtual bundles [2]. The KO-groups of spheres take the values as shown in the following table:

Dp-brane	D9	D8	D7	D6	D5	D4	D3	D2	D1	D0	D(−1)
Transverse $\mathbf{X}$	$\mathbf{S}^0$	$\mathbf{S}^1$	$\mathbf{S}^2$	$\mathbf{S}^3$	$\mathbf{S}^4$	$\mathbf{S}^5$	$\mathbf{S}^6$	$\mathbf{S}^7$	$\mathbf{S}^8$	$\mathbf{S}^9$	$\mathbf{S}^{10}$
$\widetilde{\mathrm{KO}}(\mathbf{X})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 2: KO-groups of spheres.

The values for $\widetilde{\mathrm{KO}}(\mathbf{S}^n)$ reproduce the known BPS Dp-branes of ten-dimensional Type I string theory, namely those with $p = 1, 5, 9$. In addition, we encounter $\mathbb{Z}_2$-charged non-BPS Dp-branes with $p = -1, 0, 7$ and 8. The former carry RR charge and correspond to boundary states of the form

$$|Dp\rangle = \frac{1}{\sqrt{2}}(|Bp\rangle_{NSNS} \pm |Bp\rangle_{RR})\,, \tag{3.1}$$

where the relative sign differentiates brane from antibrane, whereas the latter do not carry RR charge, and therefore correspond to states of the form

$$|Dp\rangle = |Bp\rangle_{NSNS}\,, \tag{3.2}$$

and are their own antibranes. All the properties of the non-BPS D-branes can be obtained from these boundary states via tree-level overlaps with other boundary states [15, 4, 1] (for an extension to higher loops, see [2]). This construction proves that all charges from the previous table are carried by D-branes, i.e. spacetime defects on which strings can end.

The most useful description of the non-BPS D-branes is often in terms of bound states of a *single* BPS D-brane D-antibrane pair with lowest possible dimension. In this approach, the D0-brane and D8-brane simply correspond to topologically stable kinks in the tachyon (Higgs) field living on the worldvolume of the D1-$\overline{\text{D1}}$ and D9-$\overline{\text{D9}}$ systems, respectively [1]. The D(−1)-brane and D7-brane, on the other hand, correspond to the D(−1)-$\overline{\text{D}(-1)}$ and D7-$\overline{\text{D7}}$ systems in Type IIB, respectively, projected by Ω [2]. This approach allows one to easily deduce the worldvolume theories of the non-BPS D-branes, and in particular the worldvolume gauge groups as in table 3:

Dp-brane	D0	D1	D5	D7	D8	D9
Gauge group	$\mathbb{Z}_2$	$\mathbb{Z}_2$	USp(2)	U(1)	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 3: Gauge groups.

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3.2 $D = 9$

As in the Type II case, D-brane charges of Type I compactified on a circle are classified by the relative K-theory group $\mathrm{KO}(X \times \mathbf{S}^1,\, \mathbf{S}^1)$. This group is evaluated in the appendix, giving:

$$\mathrm{KO}(X \times \mathbf{S}^1, \mathbf{S}^1) = \widetilde{\mathrm{KO}}^{-1}(X) \oplus \widetilde{\mathrm{KO}}(X)\,. \tag{3.3}$$

Thus, we obtain the nine-dimensional stable D-brane charge spectrum as table 4:

Dp-brane	D8	D7	D6	D5	D4	D3	D2	D1	D0	D(−1)
Transverse $\mathbf{X}$	$\mathbf{S}^0$	$\mathbf{S}^1$	$\mathbf{S}^2$	$\mathbf{S}^3$	$\mathbf{S}^4$	$\mathbf{S}^5$	$\mathbf{S}^6$	$\mathbf{S}^7$	$\mathbf{S}^8$	$\mathbf{S}^9$
$\widetilde{\mathrm{KO}}(\mathbf{X})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$\widetilde{\mathrm{KO}}^{-1}(\mathbf{X})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 4: $D = 9$ stable brane charges.

Note that the relative K-theory groups correctly include the nine-dimensional Dp-brane charges that correspond to unwrapped Dp-branes as well as wrapped D$(p+1)$-branes in the ten-dimensional theory.

Under T-duality, Type I string theory is mapped to an orientifold of Type IIA of the form $\mathbb{R}^9 \times \mathbf{S}^1/\Omega \cdot \mathcal{I}$, known as Type IA (or Type I′); here Ω acts as a reflection on the worldsheet, and $\mathcal{I}$ acts as a reflection on the compact direction. The compact direction is therefore an interval, rather than a circle. The associated relative K-theory group is given by $\mathrm{KR}^{-1}(X \times \mathbf{S}^{1,1},\, \mathbf{S}^{1,1})$ [2, 3].[2] (This follows directly from the action of the orientifold group on the system of unstable D9-branes of Type IIA theory, as was briefly pointed out in [3].) We show in the appendix that the Type IA K-group decomposes as follows:

$$\mathrm{KR}^{-1}(X \times \mathbf{S}^{1,1},\, \mathbf{S}^{1,1}) = \widetilde{\mathrm{KO}}(X) \oplus \widetilde{\mathrm{KO}}^{-1}(X)\,. \tag{3.4}$$

Therefore, as in the Type II case, T-duality between Type I and Type IA theory manifests itself as an isomorphism between the relative K-theory groups,

$$\mathrm{KO}(X \times \mathbf{S}^1,\, \mathbf{S}^1) \cong \mathrm{KR}^{-1}(X \times \mathbf{S}^{1,1},\, \mathbf{S}^{1,1})\,, \tag{3.5}$$

whose elements correspond to D-brane charges of the orientifold compactification.

The isomorphism (3.5) again maps the first term of the relative KO-group in (3.3) to the second term of the relative KR-group in (3.4), and vice versa. It is therefore tempting to identify the respective terms as the contributions to nine-dimensional D-brane charges coming from unwrapped and wrapped ten-dimensional D-branes.

[2] Here we use standard mathematical parlance [16] to denote by $\mathbf{S}^{1,1}$ the unit circle inside the plane $\mathbb{R}^{1,1}$ where the KR involution leaves the first coordinate invariant and reflects the second (cf. the appendix).

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For example, nine-dimensional 0-brane charge in Type I receives a $\mathbb{Z}_2$ contribution from the unwrapped non-BPS D0-brane, and a $\mathbb{Z}$ contribution from wrapped BPS D1-branes. In the Type IA description, it receives contributions from the T-dual configurations, i.e. $\mathbb{Z}_2$ from the wrapped non-BPS D1-brane, and $\mathbb{Z}$ from unwrapped BPS D0-branes. However, on the face of it, there seems to be a problem with this interpretation: the non-BPS D-branes are not stable for all radii, and therefore cannot contribute conserved charges everywhere in moduli space. What is then responsible for the $\mathbb{Z}_2$ charges when the non-BPS D-branes are unstable? To answer this question, let us first recall how non-BPS D-branes decay.

3.3 D-brane decay

Consider for example the non-BPS D0-brane in Type I. The spectrum of open strings beginning and ending on the D0-brane is tachyon-free in ten dimensions. Once we compactify on a circle however, the ground state at winding number 1 will have a classical mass squared given by ($\alpha' = 1$)

$$m^2 = -\frac{1}{2} + R^2 \,, \tag{3.6}$$

and will therefore become tachyonic when $R < 1/\sqrt{2}$. As a result, the D0-brane should then decay into a wrapped D1-D$\overline{1}$ system. Recall, however, that in describing the D0-brane as a D1-D$\overline{1}$ bound state one requires the tachyon (Higgs) field of the D1-D$\overline{1}$ system to condense into a kink. This implies anti-periodic boundary conditions for the tachyon, achieved by turning on a $\mathbb{Z}_2$ Wilson line on either the D1-brane or the D$\overline{1}$-brane. The $\mathbb{Z}_2$ charge of the D0-brane is not lost when it decays, but rather reappears as a $\mathbb{Z}_2$-valued Wilson line on its decay products. The unwrapped D8-brane and D7-brane meet a similar fate when $R < 1/\sqrt{2}$; they decay into a wrapped D9-D$\overline{9}$ and D8-D8 system, respectively, also with a non-trivial $\mathbb{Z}_2$ Wilson line.

Conversely, after T-duality the above unwrapped non-BPS Dp-branes become wrapped non-BPS D$(p+1)$-branes of Type IA with $p+1 = 1, 8, 9$. These develop a tachyon of unit momentum when the dual radius becomes too large ($R' > \sqrt{2}$), and consequently decay into an unwrapped Dp-D$\overline{p}$ system restricted to the orientifold planes. The non-trivial $\mathbb{Z}_2$ Wilson line in the Type I description reflects the presence of the resulting D-brane and D-antibrane on *different* orientifold planes. We conclude that in the Type IA picture the $\mathbb{Z}_2$ charge of the decaying D-brane is encoded in the $\mathbb{Z}_2$ choice of locations for its decay products. These decay processes are summarized in fig. 1.

For the euclidean wrapped $\mathbb{Z}_2$ D0-brane and the $\mathbb{Z}_2$ D-instanton, compactification on a circle again introduces regions of "stability" on the moduli space of the circle, but since these are instantonic configurations, we now compare the values of the instanton action.

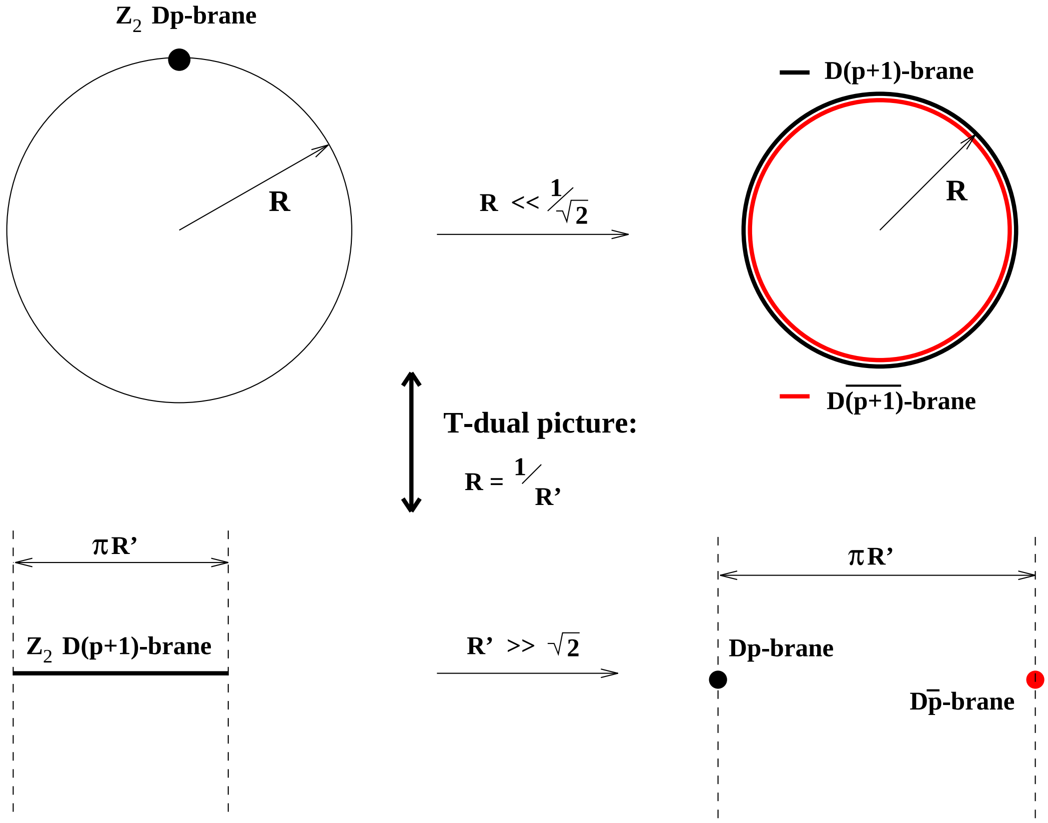

Figure 1: Decay of non-BPS D-branes in Type I and Type IA.

3.4 D-brane transfer

The picture of D-brane decay described above offers insight for the resolution of another puzzle which is most clearly illustrated in the Type IA picture. When $R' < \sqrt{2}$, there seem to be two distinct sources of $\mathbb{Z}_2$ charge associated with D0-branes in the nine-dimensional theory. The first one is due to the possibility of locating a single D0-brane at either orientifold plane, while the second is due to the stretched non-BPS D1-brane. However, K-theory indicates that there is only one D0-brane $\mathbb{Z}_2$ charge.

As we saw above, when $R' > \sqrt{2}$ the D1-brane decays into a D0-brane at one O8-plane, and a D$\overline{0}$-brane at the other O8-plane. This is a crucial clue for the resolution of our puzzle. Consider a configuration consisting of a stuck D0-brane (half D0-brane) at one orientifold plane and a wrapped non-BPS D1-brane. As far as conserved D-brane charges are concerned, this configuration is completely equivalent to a stuck D0-brane at the other orientifold plane, and in fact unstable to decay into it. The same is true for the unwrapped D7 and D8-brane. In each case, a D-brane stuck at one orientifold plane is "transferred" by a wrapped D-brane of one higher dimension to the other orientifold plane. Thus we see that the puzzle of missing $\mathbb{Z}_2$ charges in K-theory is resolved by a "brane transfer operation" (see fig. 2).

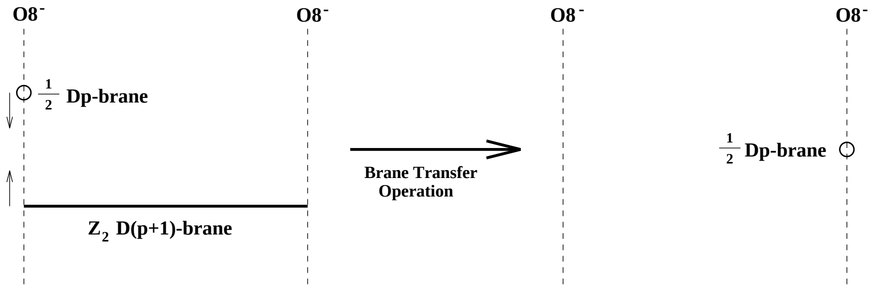

Figure 2: D-brane transfer operation in Type IA.

3.5 $D < 9$

Our analysis can be straightforwardly extended to T-duality of Type I theory on higher tori. D-brane charges in Type I theory on $\mathbf{T}^m$ correspond to the relative K-theory group (calculated by iterating (A.17))

$$\mathrm{KO}(X \times \mathbf{T}^m, \mathbf{T}^m) = \bigoplus_{p=0}^{m} \binom{m}{p} \widetilde{\mathrm{KO}}^{-p}(X)\,, \tag{3.7}$$

to be compared to the corresponding group on the T-dual side.

The T-dual orientifold theory is $\mathrm{IIA}/\Omega\cdot\mathcal{I}_m$ if m is odd, and $\mathrm{IIB}/\Omega\cdot\mathcal{I}_m$ if m is even, with $\mathcal{I}_m$ the involution (times appropriate factors of $(-1)^{F_L}$) that reflects m compact dimensions. D-brane charges in these theories will therefore be classified by the relative KR-group $\mathrm{KR}^{-n}(X \times \mathbf{T}^m, \mathbf{T}^m)$ (for some n), where the involution of KR-theory acts trivially on X, and reflects all the dimensions of $\mathbf{T}^m$. An interesting subtlety arises when we try to determine the value of n that corresponds to $\mathbf{T}^m$. In Type IA theory, i.e. for $m = 1$, one could use the alternative string theory definition of $\mathrm{KR}^{-1}(X)$ to demonstrate that the appropriate value of n is in fact $n = 1$ [3]. For $m > 1$, however, we do not seem to have that option. Instead, we will proceed as follows.

Agreement with the known spectrum of *supersymmetric* D-branes determines that $n = m$ mod 4. Since the Bott periodicity of KO-theory is eight, this leaves an uncertainty as to whether n equals m or $m + 4$. We claim that the correct value is $n = m$, and the D-brane charges in the Type I T-dual models are classified by the relative group

$$\mathrm{KR}^{-m}(X \times \mathbf{T}^m, \mathbf{T}^m)\,. \tag{3.8}$$

Indeed, iterating (A.23), and using the $(1, 1)$ Bott periodicity of KR-theory, one can show that

$$\mathrm{KR}^{-n}(X \times \mathbf{T}^m, \mathbf{T}^m) = \bigoplus_{p=0}^{m} \binom{m}{p} \widetilde{\mathrm{KR}}^{n,p}(X) = \bigoplus_{p=0}^{m} \binom{m}{p} \widetilde{\mathrm{KO}}^{p-n}(X)\,; \tag{3.9}$$

this coincides with (3.7) for $n = m$.[3] Thus, we again get a T-duality isomorphism

$$\mathrm{KR}^{-m}(X \times \mathbf{T}^m, \mathbf{T}^m) \cong \mathrm{KO}(X \times \mathbf{T}^m, \mathbf{T}^m)\,. \tag{3.10}$$

The corresponding spectrum of both the BPS and the ($\mathbb{Z}_2$) stable non-BPS D-branes is in precise agreement with the degeneracies of various wrapped branes. Still, the precise bookkeeping of D-brane charges will involve various higher-dimensional analogs of the brane transfer operation studied above. As a particularly interesting example, consider the classification of D-instantons in Type IIB on $T^2/\Omega \cdot \mathcal{I}_2$, in terms of $\mathrm{KR}^{-2}(\mathbf{S}^8 \times \mathbf{T}^2,\, \mathbf{T}^2) = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This group contains fewer charges than naively expected; this is again resolved by brane transfer operations, which now involve objects that wrap zero, one, or both dimensions of the compact $\mathbf{T}^2$. We leave details as an exercise to the interested reader.

4. Type $\widetilde{\mathrm{I}}$ Theories

When considering theories in nine dimensions, there exists a natural extension of Type IA which involves replacing one of its two $\mathrm{O8}^-$ planes with an $\mathrm{O8}^+$ plane [17]. We will refer to it as Type $\widetilde{\mathrm{IA}}$ theory. This theory requires no D8-branes, and so has no gauge group, yet it still contains interesting non-BPS stable D-branes in its spectrum. We will demonstrate that some charges which are locally stable at one type of orientifold plane will now become unstable due to the possibility of moving to another type of orientifold plane and annihilating there. Also, stretched $\mathbb{Z}_2$ charged D-branes will no longer be connected with the transfer of stuck branes from one plane to the other.

To classify the possible D-brane charges of Type $\widetilde{\mathrm{IA}}$ we will once again analyze possible tachyon backgrounds of unstable D9-branes using K-theory. For this purpose, it is easier to start with the T-dual of Type $\widetilde{\mathrm{IA}}$, Type $\widetilde{\mathrm{I}}$ theory. This T-dual was worked out in [18], and consists of gauging a $\mathbb{Z}_2$ symmetry of IIB on a circle which is realized by composing worldsheet parity reversal (Ω) with a half-circumference shift along the circle. The natural K-group of D-brane charges for Type $\widetilde{\mathrm{I}}$ is then the relative group $\mathrm{KR}(X \times \mathbf{S}^{0,2},\, \mathbf{S}^{0,2})$.[4] Since $\mathbf{S}^{0,2}$ is just a circle in $\mathbb{R}^2$ with both dimensions reflected, the involution of KR-theory indeed acts on $\mathbf{S}^{0,2}$ by the required shift.

The Type $\widetilde{\mathrm{I}}$ K-group, $\mathrm{KR}(X \times \mathbf{S}^{0,2})$, is known in the mathematics literature to be isomorphic to the K-group $\mathrm{KSC}(X)$ of self-conjugate bundles on X (see [16, 19], [20] and the appendix). We bring this up because $\mathrm{KSC}(X)$ has several nice features.

[3] Heuristically, the exponent $-m$ in the relevant KR-group has to do with the fact that $\mathbf{S}^1$ with the reflection $\mathcal{I}$ behaves effectively as a "sphere of dimension minus one" in KR-theory: while the wedge product with $\mathbf{S}^1$ with trivial involution lowers the exponent of the KR-group by one, the wedge product with $\mathbf{S}^1$ with the reflection raises the exponent by one; cf. the appendix.

[4] We are again using the $\mathbf{S}^{p,q}$ notation reviewed in the appendix.

Dp-brane	D8	D7	D6	D5	D4	D3	D2	D1	D0	D(-1)
$\widetilde{\mathrm{KSC}}(\mathbf{S}^{8-p})$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 5: Spectrum of $D = 9$ Type $\widetilde{\mathrm{I}}$ and $\widetilde{\mathrm{IA}}$ D-branes.

First of all, using the relation to KSC-theory, we can show for the relative groups that

$$\mathrm{KR}(X \times \mathbf{S}^{0,2}, \mathbf{S}^{0,2}) \cong \mathrm{KR}^{-4}(X \times \mathbf{S}^{0,2}, \mathbf{S}^{0,2}) . \tag{4.1}$$

The appearance of $\mathrm{KR}^{-4}(X \times \mathbf{S}^{0,2})$ is very interesting here, as this group associates a symplectic projection to Ω. Thus, the period of four indicated by (4.1) fits nicely with having both an $\mathrm{O8}^-$ plane and $\mathrm{O8}^+$ plane in Type $\widetilde{\mathrm{IA}}$ — indeed, it means that orthogonal and symplectic groups appear on the same footing in this model.

Second, KSC groups have been calculated for all spheres $\mathbf{S}^n$ [19], which means we can immediately read off the complete nine-dimensional spectrum of Type $\widetilde{\mathrm{I}}$ and $\widetilde{\mathrm{IA}}$ D-branes in table 5. (Note that D8-branes appear on this list, even though the tadpole cancellation argument will restrict the net number of D8-branes in the Type $\widetilde{\mathrm{IA}}$ vacuum to zero.)

Unlike the relative K-theory groups that appeared in the previous sections, $\mathrm{KSC}(X)$ does not naturally split into subgroups related to wrapped and unwrapped branes. Undeterred, we will try to analyze the physical spectrum listed above in terms of the wrapped and unwrapped Dp-branes of Type $\widetilde{\mathrm{IA}}$ string theory, hoping to learn an interesting lesson when this strategy becomes inadequate.

4.1 Unwrapped D-branes of Type $\widetilde{\mathrm{IA}}$

Our strategy for determining which nine-dimensional Dp-branes come from unwrapped D-branes of Type $\widetilde{\mathrm{IA}}$ (which are point-like along the interval) is to use our knowledge of Type IA theory to list the stable D-brane spectrum near an $\mathrm{O8}^-$ plane, and to use a simple period shift to list the stable D-brane spectrum near an $\mathrm{O8}^+$ plane (cf. [14]). Modulo some identifications, this gives all the D-branes in the previous table, which come from unwrapped branes of Type $\widetilde{\mathrm{IA}}$. It will be instructive to follow this piecewise analysis of the compactification manifold; its eventual inability to explain the detailed spectrum of non-BPS states leads to interesting conclusions.

In the vicinity of the $\mathrm{O8}^-$ plane, the D-brane spectrum is given as in table 6:

Dp-brane	D8	D7	D6	D5	D4	D3	D2	D1	D0	D(-1)
$\widetilde{\mathrm{KO}}(\mathbf{S}^{8-p})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 6: Spectrum near the $\mathrm{O8}^-$ plane.

The O8$^+$ plane differs from O8$^-$ by interchanging SO and Sp projections. Due to Bott periodicity between KO- and KSp-theory, the switch from O8$^-$ to O8$^+$ corresponds to swapping Dp-branes with D$(p+4)$-branes, and leads to the spectrum near the O8$^+$ plane:

Dp-brane	D8	D7	D6	D5	D4	D3	D2	D1	D0	D(-1)
$\widetilde{\mathrm{KSp}}(\mathbf{S}^{8-p})$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0

Table 7: Spectrum near the O8$^+$ plane.

Now we need only combine the last two tables and compare with table 5.

Looking first at the $\mathbb{Z}$–valued D-brane charges, we see that we can correctly account for the BPS D0-branes and D4-branes of Type $\widetilde{\mathrm{IA}}$. The fact that these appear in both table 6 and table 7 reflects the fact that two half-D0-branes on the O8$^-$ plane can combine to make a single D0-brane in the bulk which then becomes a D0-brane on the O8$^+$ plane, and similarly for the half-D4-branes on the O8$^+$ plane. Since half-D-branes are now limited to living on only one of the O8 planes, there is no need of extra $\mathbb{Z}_2$ charges for brane transfer operations.

Having been successful with the BPS D-branes that carry conventional RR charges, we now turn to the $\mathbb{Z}_2$-charged non-BPS D-branes of tables 6 and 7. Here we run into an interesting puzzle: while these tables correctly account for the $\mathbb{Z}_2$-charged D(-1)-brane, D3-brane and D7-brane of table 5, they also predict a $\mathbb{Z}_2$-charged D2-brane and D6-brane, which are however absent in table 5. The resolution of this puzzle reveals an interesting new effect. Take, for example, the stable $\mathbb{Z}_2$-charged non-BPS D6-brane identified in table 6 near the O8$^-$ plane. It consists of a D6-brane and its mirror D$\overline{6}$-brane, where the usual tachyon between the two has been removed by the orientifold projection. Just like the Type I non-BPS D7-brane in section 3, this system carries a U(1) gauge group, and can separate in a symmetric fashion and transfer over to the other O8 plane. In Type $\widetilde{\mathrm{IA}}$, however, the orientifold projection is different at the other O8 plane, and therefore the tachyon is no longer removed. This implies that the non-BPS D6-brane locally stable near the O8$^-$ plane is no longer stable in the global theory.

This effect has important consequences for the analysis of non-BPS D-branes on compact manifolds. Typically, when looking at the BPS spectrum of D-branes near singularities of a compact manifold such as K3 orientifolds, one can look piecewise at the singularities (i.e. approximate them with ALE spaces) and add the corresponding spectra (being careful to match the bulk D-branes). We now see that for stable non-BPS D-branes this is a dangerous procedure, as D-branes locally stable near one kind of singularity can become unstable due to other singularities in the complete space.

Now that we have successfully accounted for the unwrapped D-branes, we can move on to look at the charges in table 5 which come from wrapped D-branes of Type $\widetilde{\mathrm{IA}}$.

4.2 Wrapped D-Branes of Type $\widetilde{\mathrm{IA}}$

To examine how the wrapped D-branes of Type $\widetilde{\mathrm{IA}}$ contribute to the nine-dimensional spectrum listed in table 5, it is more convenient to shift to the T-dual Type $\widetilde{\mathrm{I}}$ point of view. We can now use a simple construction to build their dual Type $\widetilde{\mathrm{I}}$ D-branes, which are now unwrapped. The IIB $\mathbb{Z}_2$ symmetry we gauged to get Type $\widetilde{\mathrm{I}}$ strings included a half-shift along a circle. Requiring that we respect the $\mathbb{Z}_2$ symmetry means we match each unwrapped Dp-brane with another Dp-brane at the opposite position along the circle for $p = 1$ or 5, and with a D$\overline{p}$-brane for $p = -1$, 3, or 7. The first configuration is BPS, and correspondingly yields the stable $\mathbb{Z}$-charged D1-brane and D5 brane in table 5. Note that these D-branes will carry a $\mathrm{U}(N)$ gauge group, and will correspond to doubly wrapped BPS D2-branes and D6-branes in the Type $\widetilde{\mathrm{IA}}$ theory.

The second kind of configuration above is more interesting, as it yields stable non-BPS D-branes. It is clear that these states carry a $\mathbb{Z}_2$ charge, since when two of them are present there is an allowed motion which enables the D-branes and D-antibranes to annihilate, as can be seen in fig. 3. Consequently, there appear to be two sources of $\mathbb{Z}_2$ p-brane charge in Type $\widetilde{\mathrm{I}}$ for $p = -1, 3, 7$; the first is due to wrapped $(p+1)$-branes (the T-duals of the unwrapped Type $\widetilde{\mathrm{IA}}$ D-branes discussed in the previous subsection), and the second is due to the above $p - \overline{p}$ combinations. There is no contradiction with the fact that K-theory predicts only a single $\mathbb{Z}_2$ (table 5) since the above states are stable in complementary regions of moduli space (fig. 4).

5. Conclusions

In this paper, we used the picture of stable D-branes as topological defects in unstable brane systems to study the charges of stable non-BPS D-branes in string compactifications. K-theory turns out to be a useful tool in this pursuit.

We have seen that T-duality appears to be a manifest symmetry in K-theory. In Type II and Type I on $\mathbf{T}^m$, we have identified the relative K-theory groups on both sides of T-duality, and have demonstrated that they are isomorphic. In the slightly more exotic Type $\widetilde{\mathrm{I}}$ Type $\widetilde{\mathrm{IA}}$ T-duality, we have identified the K-theory group of D-brane charges only on the Type $\widetilde{\mathrm{I}}$ side, and then demonstrated that it exactly corresponds to D-brane charges on the Type $\widetilde{\mathrm{IA}}$ side. Of course, one should be able to identify the K-groups on *both* sides of T-duality from first principles, by studying how the orientifold group acts on unstable systems of spacetime-filling branes. We certainly expect that such a direct analysis will confirm our findings, and will provide an extra check that T-duality is a manifest symmetry in K-theory. It would also be instructive to extend our analysis of Type $\widetilde{\mathrm{I}}$ to all Type I models without vector structure [18].

In the process of identifying the D-branes which carry the charges predicted by K-theory, we have come across several interesting effects. An apparent abundance of some $\mathbb{Z}_2$ charges in the string theory construction is resolved by brane transfer

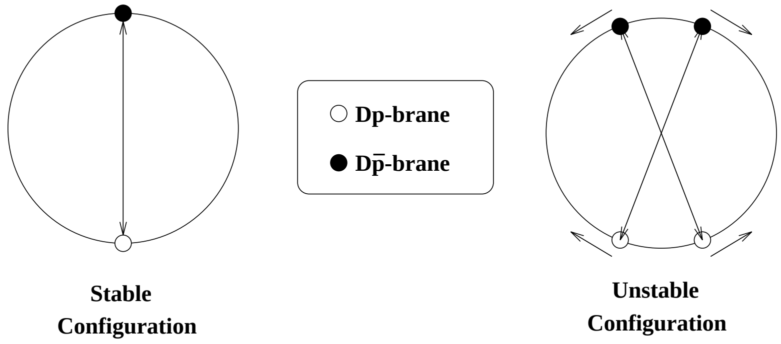

Figure 3: $\mathbb{Z}_2$ annihilation of D-branes in Type $\widetilde{\text{I}}$.

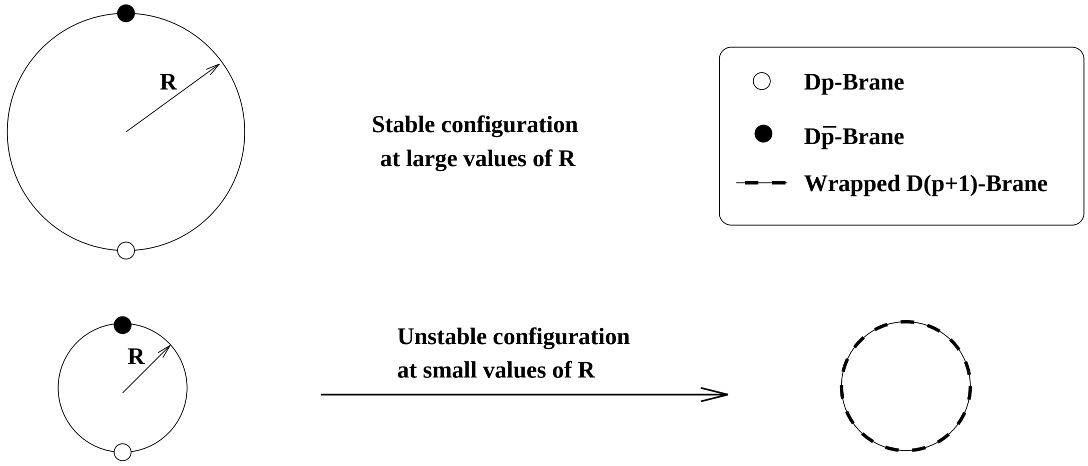

Figure 4: Instability of the Dp-brane D$\overline{p}$-brane system.

operations. Other $\mathbb{Z}_2$ charges, apparently conserved locally near an orientifold plane, dissipate in the full theory due to the presence of another O-plane. We believe that these phenomena occur in a more general class of compactifications, indicating that the piecewise analysis of stable non-BPS D-brane spectra should only be trusted when these phenomena are taken into account.

T-duality is to be contrasted with other string theory dualities, such as S-duality of Type IIB string theory; whether or not there is an extension of K-theory that incorporates Type IIB S-duality — and in particular explains NS states — remains one of the many intriguing open questions of this framework. (For a possible step in this direction, see [21].)

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A. Useful facts in K-theory

In this somewhat extensive appendix, we present a summary of some basic notions of K-theory (and its connection to string theory), as well as technical details of various K-theory calculations needed for some of our arguments in the body of the paper. For a more comprehensive introduction to K-theory, the reader should consult [22, 23, 24, 25]. Some elementary K-theory facts that arise in the string theory context can also be found in [2, 3].

Unitary K-theory $\mathrm{K}^{-n}(X)$ The unitary K-theory group $\mathrm{K}(X)$ is defined, for a given compact manifold X,[5] as the group of equivalence classes of pairs of unitary bundles (E, F) on X, where two pairs are declared equivalent if they can be made isomorphic to each other by adding pairs of isomorphic bundles (H, H).

The reduced group $\widetilde{\mathrm{K}}(X)$ is defined as the kernel of the natural map $\mathrm{K}(X) \rightarrow \mathrm{K}(x_0)$ to the K-theory group $\mathrm{K}(x_0) = \mathbb{Z}$ of a point x_0 in X, induced from the map $x_0 \rightarrow X$. The full K-theory group splits canonically as $\mathrm{K}(X) = \widetilde{\mathrm{K}}(X) \oplus \mathbb{Z}$. In Type IIB string theory on X, E and F are the Chan-Paton bundles of branes and anti-branes wrapping X, and $\widetilde{\mathrm{K}}(X)$ classifies invariant charges that can be carried by general tadpole-cancelling configurations of brane-antibrane pairs, modulo pair creation and annihilation.

Higher (reduced) K-theory groups $\widetilde{\mathrm{K}}^{-n}(X)$ are defined by

$$\widetilde{\mathrm{K}}^{-n}(X) = \widetilde{\mathrm{K}}(X \wedge \mathbf{S}^n)\,. \tag{A.1}$$

Here the wedge product is defined for two manifolds X and Y with a marked point x_0 in X and y_0 in Y, as the (topological) coset[6]

$$X \wedge Y = X \times Y / (X \times y_0) \cup (Y \times x_0)\,. \tag{A.2}$$

(For example, $\mathbf{S}^1 \wedge \mathbf{S}^1 = \mathbf{S}^2$, $\mathbf{S}^n \wedge \mathbf{S}^1 = \mathbf{S}^{n+1}$.) The higher reduced groups are related to the unreduced groups by $\mathrm{K}^{-n}(X) = \widetilde{\mathrm{K}}^{-n}(X) \oplus \mathrm{K}^{-n}(x_0)$, where the K-theory groups of a point are given by

$$\begin{aligned} \mathrm{K}^{-2p}(x_0) &= \mathbb{Z}\,, \\ \mathrm{K}^{-2p-1}(x_0) &= 0\,. \end{aligned} \tag{A.3}$$

Therefore $\mathrm{K}(X) = \widetilde{\mathrm{K}}(X) \oplus \mathbb{Z}$ and $\mathrm{K}^{-1}(X) = \widetilde{\mathrm{K}}^{-1}(X)$.

[5] Unless explicitly stated otherwise, X in this paper is always a compact connected manifold. K-theory can be extended to non-compact manifolds, as K-theory with compact support; one essentially defines $\mathrm{K}(Z) = \widetilde{\mathrm{K}}(\widetilde{Z})$, where Z is the one-point compactification of Z. Thus, K-theory with compact support can be related to K-theory of compact manifolds, and we will not use it explicitly in this paper.

[6] Given a closed submanifold Y in a compact manifold X, the topological coset X/Y is defined by shrinking Y — as a subset in X — into a point.

The definition of the higher K-theory group $\mathrm{K}^{-1}(X)$ (which classifies Type IIA theory D-branes) in terms of the $p+1$-dimensional extension $X \wedge \mathbf{S}^1$ of the p-dimensional manifold X is rather awkward for string theory purposes, as it invokes an extra spacetime dimension $\mathbf{S}^1$. There is an alternative definition of $\mathrm{K}^{-1}(X)$, in terms of pairs (E,α) where E is a bundle on X and α is an automorphism on E (see, e.g., [22], II.3.3). It is, in fact, this alternative definition that appears directly from the worldvolume dynamics of spacetime-filling unstable D9-branes in Type IIA theory [3].

K-theory is intimately related to homotopy theory. K-theory groups of any compact manifold X can be understood as the groups of homotopy classes of maps from X to certain classifying spaces,

$$\begin{aligned} \widetilde{\mathrm{K}}(X) &= [X, BU]\,, \\ \mathrm{K}^{-1}(X) &= [X, U]\,. \end{aligned} \tag{A.4}$$

Here U and BU are the $N\to\infty$ limits of the unitary group $\mathrm{U}(N)$ and the (group) coset $\mathrm{U}(2N)/\mathrm{U}(N)\times\mathrm{U}(N)$, respectively.[7]

Given a closed submanifold Y in a compact manifold X, one defines the relative K-theory group $\mathrm{K}(X,\,Y)$ as follows. Just like in the definition of $\mathrm{K}(X)$, we start with a pair of bundles (E,F) on X. In addition, we choose a "trivialization" along Y, i.e. an isomorphism $\alpha : E|_Y \to F|_Y$ between the restrictions of E and F to the submanifold Y. One defines a certain equivalence relation on such triples (E,F,α), declaring two such triples equivalent if they can be made isomorphic by creation or annihilation of triples (H,H,id_H) (see [22], II.2.29 for the mathematicians' definition). Similarly, higher relative K-groups are defined by

$$\mathrm{K}^{-n}(X,\,Y) = \mathrm{K}(X\times\mathbf{B}^n, X\times\mathbf{S}^{n-1}\cup Y\times\mathbf{B}^n)\,, \tag{A.5}$$

where $\mathbf{B}^n$ is the unit ball in $\mathbb{R}^n$, and $\mathbf{S}^{n-1}$ is its boundary sphere. The relative groups (A.5) represent a generalization of the reduced groups $\widetilde{\mathrm{K}}^{-n}(X)$, since one can write $\widetilde{\mathrm{K}}^{-n}(X) = \mathrm{K}^{-n}(X,x_0)$ with x_0 a point in X. Also, relative groups are related to the reduced groups by $\mathrm{K}^{-n}(X,Y) = \widetilde{\mathrm{K}}^{-n}(X/Y)$ whenever both sides of this equation make sense.

The relative groups $\mathrm{K}^{-n}(X,\,Y)$ are important because they connect the groups of X and Y via the exact sequence

$$\mathrm{K}^{-n-1}(Y)\to\mathrm{K}^{-n-1}(X)\to\mathrm{K}^{-n}(X,\,Y)\to\mathrm{K}^{-n}(Y)\to\mathrm{K}^{-n}(X) \tag{A.6}$$

(valid for any $n\geq 0$), which is reminiscent of similar exact sequences from cohomology theory. In fact, K-theory is a *generalized* cohomology theory — it satisfies all the axioms of cohomology theory *except* for the dimension axiom.

[7] More generally, $\widetilde{\mathrm{K}}^{-n}(X) = [X,\Omega^n BU]$, where $\Omega^n Y$ is the n-th iterated loop space of Y. One can prove that ΩBU is homotopically equivalent to U, and $\Omega^2 BU$ is homotopically equivalent to BU. In conjunction with (A.4), this fact leads to Bott periodicity, $\mathrm{K}^{-n-2}(X) = K^{-n}(X)$.

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In the case of relative K-theory groups $\mathrm{K}(W, Y)$ that appear in this paper, the pairs W, Y are of a very special type, with $W = X \times Y$ for some manifold Y. For such pairs (or more generally, whenever W is a "retract" of Y), (A.6) can be reduced to the following split exact sequence (cf. [22], II.2.29),

$$0 \to \mathrm{K}^{-n}(X \times Y,\, Y) \to \mathrm{K}^{-n}(X \times Y) \to \mathrm{K}^{-n}(Y) \to 0\,, \tag{A.7}$$

thus leading to

$$\mathrm{K}^{-n}(X \times Y) = \mathrm{K}^{-n}(X \times Y,\, Y) \oplus \mathrm{K}^{-n}(Y)\,. \tag{A.8}$$

This formula allows one to evaluate the relative group $\mathrm{K}^{-n}(X \times Y,\, Y)$ once $\mathrm{K}^{-n}(Y)$ and $\mathrm{K}^{-n}(X \times Y)$ are found.

So far we have reduced the calculation of the relative K-theory group $\mathrm{K}^{-n}(W \times Y,\, Y)$ to the calculation of the K-theory groups of $W \times Y$. The latter can be expressed through K-theory groups of X and Y with the use of the following formula ([25], 2.4.8),

$$\widetilde{\mathrm{K}}^{-n}(X \times Y) = \widetilde{\mathrm{K}}^{-n}(X \wedge Y) \oplus \widetilde{\mathrm{K}}^{-n}(X) \oplus \widetilde{\mathrm{K}}^{-n}(Y)\,. \tag{A.9}$$

The case of our primary interest in section 2 is $Y = \mathbf{S}^1$. Using (A.9) together with $\widetilde{\mathrm{K}}^{-n}(X \wedge \mathbf{S}^1) = \widetilde{\mathrm{K}}^{-n-1}(X)$ and Bott periodicity, we get

$$\begin{aligned} \widetilde{\mathrm{K}}(X \times \mathbf{S}^1) \quad &= \mathrm{K}^{-1}(X) \oplus \widetilde{\mathrm{K}}(X), \\ \mathrm{K}^{-1}(X \times \mathbf{S}^1) &= \widetilde{\mathrm{K}}(X) \oplus \mathrm{K}^{-1}(X) \oplus \mathbb{Z}. \end{aligned} \tag{A.10}$$

This allows us to determine $\mathrm{K}(X \times \mathbf{S}^1,\, \mathbf{S}^1)$ and $\mathrm{K}^{-1}(X \times \mathbf{S}^1,\, \mathbf{S}^1)$ using (A.8), leading to (2.3) and (2.4).

Alternatively, we can calculate $\mathrm{K}^{-n}(X \times \mathbf{S}^1)$ in a manner that keeps track of the multiplicative structure of the theory. Define $\mathrm{K}^{\#}(X) = \mathrm{K}(X) \oplus \mathrm{K}^{-1}(X)$. $\mathrm{K}^{\#}(X)$ is a graded ring, with the obvious $\mathbb{Z}_2$ graded structure. We have a K-theory analog of the Künneth formula,

$$\mathrm{K}^{\#}(X \times Y) = \mathrm{K}^{\#}(X) \otimes \mathrm{K}^{\#}(Y)\,, \tag{A.11}$$

which is valid if either $\mathrm{K}^{\#}(X)$ or $\mathrm{K}^{\#}(Y)$ is freely generated (see, e.g., [22], Proposition IV.3.24). Since $\mathbf{S}^1$ has freely generated K-theory groups, we can set $Y = \mathbf{S}^1$. (Notice that this strategy for calculating K-groups of $X \times \mathbf{S}^1$ would not work in the case of KO-theory relevant for Type I, as the KO-groups of $\mathbf{S}^1$ are not freely generated.) From (A.11) we get

$$\begin{aligned} \mathrm{K}(X \times \mathbf{S}^1) &= (\mathrm{K}(X) \otimes \mathrm{K}(\mathbf{S}^1)) \oplus (\mathrm{K}^{-1}(X) \otimes \mathrm{K}^{-1}(\mathbf{S}^1)), \\ \mathrm{K}^{-1}(X \times \mathbf{S}^1) &= (\mathrm{K}(X) \otimes \mathrm{K}^{-1}(\mathbf{S}^1)) \oplus (\mathrm{K}^{-1}(X) \otimes \mathrm{K}(\mathbf{S}^1)). \end{aligned} \tag{A.12}$$

Using $\mathrm{K}(\mathbf{S}^1) = \mathbb{Z}$ and $\mathrm{K}^{-1}(\mathbf{S}^1) = \mathbb{Z}$, we again recover (A.10), which is instrumental in our proof of T-duality between Type IIA and Type IIB theories in section 2.

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With the insight from (A.12), Type II T-duality can thus be traced back to the fact that K-theory groups of $\mathbf{S}^1$ have shortened periodicity, $\mathrm{K}^{-m}(\mathbf{S}^1) = \mathrm{K}^{-m-1}(\mathbf{S}^1) = \mathbb{Z}$. Also, using (A.12), the fact that T-duality swaps wrapped and unwrapped branes corresponds to the fact that under the isomorphism (2.5) of the K-groups, $\widetilde{\mathrm{K}}(X) \otimes \mathrm{K}(\mathbf{S}^1)$ maps to $\widetilde{\mathrm{K}}(X) \otimes \mathrm{K}^{-1}(\mathbf{S}^1)$ (and similarly for $\mathrm{K}^{-1}(X)$), with $\mathrm{K}(\mathbf{S}^1)$ factors and $\mathrm{K}^{-1}(\mathbf{S}^1)$ factors interchanged.

Orthogonal K-theory $\mathrm{KO}^{-n}(X)$ **and symplectic K-theory** $\mathrm{KSp}^{-n}(X)$ $\mathrm{KO}(X)$ is the group of virtual real bundles, defined by replacing complex bundles with real bundles in the definition of $\mathrm{K}(X)$ groups. Higher KO groups are again defined via

$$\widetilde{\mathrm{KO}}^{-m}(X) = \widetilde{\mathrm{K}}(X \wedge \mathbf{S}^m)\,. \tag{A.13}$$

Just like in the unitary case, the full KO-groups are related to the reduced groups $\widetilde{\mathrm{KO}}^{-m}(X)$ by

$$\mathrm{KO}^{-m}(X) = \widetilde{\mathrm{KO}}^{-m}(X) \oplus \mathrm{KO}^{-m}(x_0)\,, \tag{A.14}$$

with x_0 a point in X. The key to the appearance of KO-theory in the bound-state construction of Type I D-branes is again its relation to homotopy theory. Just as in the unitary case, we have $\mathrm{KO}^{-n}(X) = [X, \Omega^n BO]$,[8] where BO is defined as the large-N limit of $\mathrm{O}(2N)/\mathrm{O}(N)\times\mathrm{O}(N)$, and $\Omega BO \cong O$ can be similarly approximated by $\mathrm{O}(N)$. In this case, the statement of Bott periodicity $\mathrm{KO}^{-m}(X) = \mathrm{KO}^{-m-8}(X)$ follows from the fact that $\Omega^{m+8}BO$ is homotopically equivalent to $\Omega^m BO$.

By replacing $\mathrm{O}(N)$ with $\mathrm{Sp}(N)$, and real bundles with symplectic bundles, one can similarly define the symplectic K-theory groups $\mathrm{KSp}^{-n}(X)$. Bott periodicity can be refined to show that $\mathrm{KO}^{-n}(X) = \mathrm{KSp}^{-n-4}(X)$ for any n, which means that any calculation in KSp-theory can be done in KO-theory; therefore, we will not discuss KSp-theory separately in this appendix.

Relative K-theory groups $\mathrm{KO}^{-n}(Z, Y)$ are defined by replacing complex bundles with real bundles in the definition of $\mathrm{K}^{-n}(Z, Y)$ reviewed above. For our purposes, we will again be interested in relative groups for a special class of pairs, $\mathrm{KO}^{-n}(X\times Y, Y)$; for such pairs, one can relate the relative group to $\mathrm{KO}^{-n}(X\times Y)$ and $\mathrm{KO}^{-n}(Y)$ via the following split exact sequence,

$$0 \to \mathrm{KO}^{-n}(X\times Y, Y) \to \mathrm{KO}^{-n}(X\times Y) \to \mathrm{KO}^{-n}(Y) \to 0\,, \tag{A.15}$$

leading to

$$\mathrm{KO}^{-n}(X\times Y) = \mathrm{KO}^{-n}(X\times Y, Y) \oplus \mathrm{KO}^{-n}(Y)\,. \tag{A.16}$$

The basic formula for calculating KO groups of products is again

$$\widetilde{\mathrm{KO}}(X\times Y) = \widetilde{\mathrm{KO}}(X\wedge Y) \oplus \widetilde{\mathrm{KO}}(X) \oplus \widetilde{\mathrm{KO}}(Y)\,. \tag{A.17}$$

[8] More exactly, the classifying space of KO-theory is $BO \times \mathbb{Z}$, where the extra factor of $\mathbb{Z}$ is needed to explain $\mathrm{KO}(\mathrm{pt}) = \mathbb{Z}$ (see, e.g., [22], II.1.34).

In the case of our main interest, $Y = \mathbf{S}^1$, we obtain (using $\widetilde{\mathrm{KO}}(\mathbf{S}^1) = \mathbb{Z}_2$, and $\widetilde{\mathrm{KO}}(X \wedge \mathbf{S}^1) = \widetilde{\mathrm{KO}}^{-1}(X)$)

$$\widetilde{\mathrm{KO}}(X \times \mathbf{S}^1) = \widetilde{\mathrm{KO}}^{-1}(X) \oplus \widetilde{\mathrm{KO}}(X) \oplus \mathbb{Z}_2 \,. \tag{A.18}$$

This formula is used in section 3 on the Type I side of the proof of T-duality between D-brane charges.

Real K-theory $\mathrm{KR}^{p,q}(X)$ KR-theory (introduced, under the name of "Real K-theory," by Atiyah [16]) is a generalized theory that includes unitary K-theory, KO-theory and KSp-theory (as well as the "self-conjugate" KSC-theory to be discussed below) as special cases. The corresponding groups $\mathrm{KR}(X)$ are defined for X a manifold with a selected involution τ. Basic objects are now pairs of bundles (E, F) with antilinear involution on both E and F that commutes with τ on X. Thus, $\mathrm{KR}(X)$ would be the group of virtual bundles with involutions on X.

Just like in KO-theory, one can define higher groups $\mathrm{KR}^{-m}(X)$, by

$$\widetilde{\mathrm{KR}}^{-m}(X) = \widetilde{\mathrm{KR}}(X \wedge \mathbf{S}^m); \tag{A.19}$$

the involution τ of X is extended to the involution of $X \wedge \mathbf{S}^m$ that acts trivially on $\mathbf{S}^m$.

More generally, we can consider replacing $\mathbf{S}^m$ in (A.19) by spheres with non-trivial actions of the involution. Consider $\mathbb{R}^{p,q}$, as a real manifold of dimension $p+q$ with coordinates $(x^1, \ldots, x^p, y^1, \ldots, y^q)$, and with involution that takes $(x, y) \to (x, -y)$. Similarly, one defines $\mathbf{S}^{p,q}$ as the unit sphere (of dimension $p+q-1$) in $\mathbb{R}^{p,q}$ with respect to the flat euclidean metric.[9]

Now, we can define a two-parameter set $\mathrm{KR}^{p,q}(X)$ of higher KR-theory groups, by

$$\widetilde{\mathrm{KR}}^{p,q}(X) = \widetilde{\mathrm{KR}}(X \wedge \widetilde{\mathbb{R}}^{p,q}) \,, \tag{A.20}$$

where $\widetilde{\mathbb{R}}^{p,q}$ is the one-point compactification of $\mathbb{R}^{p,q}$ (i.e. $\widetilde{\mathbb{R}}^{p,q}$ is topologically a $p+q$-sphere). By definition, (A.19) are related to (A.20) by $\mathrm{KR}^{-m}(X) = \mathrm{KR}^{m,0}(X)$. One can define relative K-theory groups $\mathrm{KR}^{-n}(Z, Y)$, again by repeating steps used in the definition of relative K-groups in K-theory and KO-theory.

Bott periodicity in KR-theory states that $\mathrm{KR}^{p,q}(X) = \mathrm{KR}^{p+1,q+1}(X)$, and that $\mathrm{KR}^{-m}(X) = \mathrm{KR}^{-m-8}(X)$. Due to the first relation, $\mathrm{KR}^{p,q}(X)$ depends only on the difference $p-q$, and one has $\mathrm{KR}^{p,q}(X) = \mathrm{KR}^{q-p}(X)$. It is interesting to notice that in KR-theory, spheres with antipodal involutions play the role of negative-dimensional spheres.

[9] Notice that our convention for $\mathbb{R}^{p,q}$ and $\mathbf{S}^{p,q}$ coincides with that of Atiyah [16], and is opposite to that of Karoubi [22].

KR-theory is a generalization of both K-theory and KO-theory. For any given X with trivial involution, we have

$$\begin{aligned} \mathrm{K}^{-m}(X) &= \mathrm{KR}^{-m}(X \times \mathbf{S}^{0,1}), \\ \mathrm{KO}^{-m}(X) &= \mathrm{KR}^{-m}(X). \end{aligned} \tag{A.21}$$

In particular, one can derive Bott periodicity in $\mathrm{K}(X)$, $\mathrm{KO}(X)$ and $\mathrm{KSp}(X)$ from the periodicities of KR-theory.

Now we are equipped to calculate the relative group $\mathrm{KR}^{-1}(X \times \mathbf{S}^{1,1}, \mathbf{S}^{1,1})$ that classifies D-brane charges in Type IA theory. (The orientifold $\mathbb{Z}_2$ acts as a reflection on the circle $\mathbf{S}^{1,1}$, and trivially on X.) This relative group is again related to $\mathrm{KR}(X \times \mathbf{S}^{1,1})$ via

$$\mathrm{KR}^{-1}(X \times \mathbf{S}^{1,1}) = \mathrm{KR}^{-1}(X \times \mathbf{S}^{1,1}, \mathbf{S}^{1,1}) \oplus \mathrm{KR}^{-1}(\mathbf{S}^{1,1}). \tag{A.22}$$

By repeating steps already familiar from the K and KO case, one obtains

$$\begin{aligned} \widetilde{\mathrm{KR}}^{-1}(X \times \mathbf{S}^{1,1}) &= \widetilde{\mathrm{KR}}^{-1}(X \wedge \mathbf{S}^{1,1}) \oplus \widetilde{\mathrm{KR}}^{-1}(X) \oplus \mathbb{Z} \\ &= \widetilde{\mathrm{KR}}^{1,1}(X) \oplus \widetilde{\mathrm{KR}}^{-1}(X) \oplus \mathbb{Z} = \widetilde{\mathrm{KO}}(X) \oplus \widetilde{\mathrm{KO}}^{-1}(X) \oplus \mathbb{Z}, \end{aligned} \tag{A.23}$$

where we have used the $(1,1)$ periodicity of KR-theory, and the fact that $\mathrm{KR}^*(X) = \mathrm{KO}^*(X)$ for the trivial orientifold action on X. Since $\widetilde{\mathrm{KR}}^{-1}(\mathbf{S}^{1,1}) = \mathrm{KR}^{-1+1}(\mathrm{pt}) = \mathrm{KO}(\mathrm{pt}) = \mathbb{Z}$, our result (3.9) follows from (A.22) and (A.23).

Self-conjugate K-theory $\mathrm{KSC}^{-n}(X)$ Given a compact manifold X, one defines a self-conjugate bundle on X as a bundle E equipped with an antilinear automorphism $\beta : E \to E$. Self-conjugate K-theory $\mathrm{KSC}(X)$ (see [19, 20], [16] and [22] III.7.13-15) is then defined by imposing a stable equivalence relation on self-conjugate bundles (i.e. on pairs (E, β)), whereby two pairs are equivalent if their sums with a third self-conjugate bundle are isomorphic (as self-conjugate bundles). Higher KSC groups are again defined via $\widetilde{\mathrm{KSC}}^{-n}(X) = \widetilde{\mathrm{KSC}}(X \wedge \mathbf{S}^n)$. The classifying space BSC of self-conjugate K-theory is described in [19] ; KSC-groups are then related to homotopy theory via $\mathrm{KSC}(X) = [X, BSC]$.

One can prove that Bott periodicity of the self-conjugate K-theory is four. This can be shown either by a direct analysis of the homotopy properties of the classifying space [19, 20] (and showing that it is homotopically equivalent to its fourth loop space), or by proving a relation between KSC-theory and KR-theory,

$$\widetilde{\mathrm{KSC}}(X) = \widetilde{\mathrm{KR}}(X \times \mathbf{S}^{0,2}), \tag{A.24}$$

and using Bott periodicity of KR-theory (see [16] for details).

The relation (A.24) between KSC and KR groups plays a central role in our analysis of T-duality between Type $\widetilde{\mathrm{I}}$ and Type $\widetilde{\mathrm{IA}}$ in section 4.

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